Gauge Theories and the Gauge Argument

Robert Moir

The University of Western Ontario
Dept. of Philosophy

February 5, 2009

1. Gauge Theories
2. Classical Electromagnetism
3. Lagrangian Mechanics of Fields
   - Lagrangian Mechanics of Point Particles
   - Lagrangian Mechanics of Fields
   - Relativistic Field Theories
4. The Gauge Argument for a Complex Scalar Field
5. Yang-Mills Theory

1 Gauge Theories

2 Classical Electromagnetism

3 Lagrangian Mechanics of Fields
- Lagrangian Mechanics of Point Particles
- Lagrangian Mechanics of Fields
- Relativistic Field Theories

4 The Gauge Argument for a Complex Scalar Field

5 Yang-Mills Theory

Gauge Theories

The Fundamental Theories of Contemporary Physics

Gauge Theories

The Fundamental Theories of Contemporary Physics

- General Relativity — Classical Field Theory (Gravity)

Gauge Theories

The Fundamental Theories of Contemporary Physics

- General Relativity — Classical Field Theory (Gravity)
- Standard Model — Quantum Field Theories

Gauge Theories

The Fundamental Theories of Contemporary Physics

- General Relativity — Classical Field Theory (Gravity)
- Standard Model — Quantum Field Theories
    - Quantum Chromodynamics (QCD) (Strong Force)

Gauge Theories

The Fundamental Theories of Contemporary Physics

- General Relativity — Classical Field Theory (Gravity)
- Standard Model — Quantum Field Theories
  - Quantum Chromodynamics (QCD) (Strong Force)
  - GWS Electroweak Theory (EWT) (Weak and EM Forces)

Field Quantization of Classical Fields

- The quantum field theories of the Standard Model are obtained by ‘quantizing’ classical field theories.

Field Quantization of Classical Fields

- The quantum field theories of the Standard Model are obtained by ‘quantizing’ classical field theories.
- It is sometimes claimed that these classical field theories can be ‘derived’ using a kind of argument called a *gauge argument*.

Gauge Arguments

- The Lagrangians of the classical field theories are invariant under a group of transformations of the field.

Gauge Arguments

- The Lagrangians of the classical field theories are invariant under a group of transformations of the field.
- Such a symmetry (invariance) is called a *global* symmetry, since the transformation of the field that leaves the Lagrangian invariant does not depend on the location in space or spacetime.

Gauge Arguments

- The Lagrangians of the classical field theories are invariant under a group of transformations of the field.
- Such a symmetry (invariance) is called a *global* symmetry, since the transformation of the field that leaves the Lagrangian invariant does not depend on the location in space or spacetime.
- The gauge argument proceeds by making the Lagrangian invariant under 'local' transformations of the field, local in the sense of depending on the location in space or spacetime (on which the field is defined).

Gauge Transformations

- The transformations of the fields in gauge arguments do not change the physical state of the field, they are transformations in an 'internal space' of the field, in a sense to be described later.

Gauge Transformations

- The transformations of the fields in gauge arguments do not change the physical state of the field, they are transformations in an 'internal space' of the field, in a sense to be described later.
- Consequently, more than one field state in this internal space describes the same physical field.

Gauge Transformations

- The transformations of the fields in gauge arguments do not change the physical state of the field, they are transformations in an 'internal space' of the field, in a sense to be described later.
- Consequently, more than one field state in this internal space describes the same physical field.
- The different states that are physically equivalent are loosely analogous to the arbitrary choice of *gauge* in the sense of an arbitrary choice of 'length' scale, *e.g.* feet or metres, seconds or years, kelvin or rankine, *etc.*

Gauge Transformations

- The transformations of the fields in gauge arguments do not change the physical state of the field, they are transformations in an 'internal space' of the field, in a sense to be described later.
- Consequently, more than one field state in this internal space describes the same physical field.
- The different states that are physically equivalent are loosely analogous to the arbitrary choice of *gauge* in the sense of an arbitrary choice of 'length' scale, *e.g.* feet or metres, seconds or years, kelvin or rankine, *etc.*
- Consequently, the transformations leaving the Lagrangian invariant are called *gauge transformations*.

Gauge Arguments

- A gauge argument starts with a Lorentz covariant Lagrangian of a classical field that is invariant under a global gauge transformation, or gauge transformation of the first kind.

Gauge Arguments

- A gauge argument starts with a Lorentz covariant Lagrangian of a classical field that is invariant under a global gauge transformation, or gauge transformation of the first kind.
- An attempt is then made to make the Lagrangian invariant under the same kind of transformation that varies depending on the spacetime location, *i.e.* a local gauge transformation, or gauge transformation of the second kind.

Gauge Arguments

- A gauge argument starts with a Lorentz covariant Lagrangian of a classical field that is invariant under a global gauge transformation, or gauge transformation of the first kind.
- An attempt is then made to make the Lagrangian invariant under the same kind of transformation that varies depending on the spacetime location, *i.e.* a local gauge transformation, or gauge transformation of the second kind.
- The initial Lagrangian is not invariant under a local gauge transformation and the the gauge transformed Lagrangian ceases to be Lorentz covariant.

Gauge Arguments

- A gauge argument starts with a Lorentz covariant Lagrangian of a classical field that is invariant under a global gauge transformation, or gauge transformation of the first kind.
- An attempt is then made to make the Lagrangian invariant under the same kind of transformation that varies depending on the spacetime location, *i.e.* a local gauge transformation, or gauge transformation of the second kind.
- The initial Lagrangian is not invariant under a local gauge transformation and the the gauge transformed Lagrangian ceases to be Lorentz covariant.
- Terms are introduced to restore Lorentz and local gauge invariance, which introduces one or more vector fields $A_\mu(x^\nu)$.

Gauge Arguments

- A gauge argument starts with a Lorentz covariant Lagrangian of a classical field that is invariant under a global gauge transformation, or gauge transformation of the first kind.
- An attempt is then made to make the Lagrangian invariant under the same kind of transformation that varies depending on the spacetime location, *i.e.* a local gauge transformation, or gauge transformation of the second kind.
- The initial Lagrangian is not invariant under a local gauge transformation and the the gauge transformed Lagrangian ceases to be Lorentz covariant.
- Terms are introduced to restore Lorentz and local gauge invariance, which introduces one or more vector fields $A_\mu(x^\nu)$.
- Then allowing A_μ to contribute directly to the Lagrangian introduces a gauge field tensor $F_{\mu\nu}$ for each A_μ.

Gauge Arguments

- By choosing certain symmetry groups under which the Lagrangian is invariant, it is claimed that new physical fields A_μ that interact with the fields you begin with are introduced.

Gauge Arguments

- By choosing certain symmetry groups under which the Lagrangian is invariant, it is claimed that new physical fields A_μ that interact with the fields you begin with are introduced.
- Beginning with a field ϕ, by choosing the symmetry group to be $U(1)$, ϕ picks up an interation with the classical electromagnetic field, for $SU(3)$ ϕ picks up an interaction with a 'colour field,' and for $U(1) \otimes SU(2)$ ϕ picks up an interaction with an 'electroweak field.'

Gauge Arguments

- By choosing certain symmetry groups under which the Lagrangian is invariant, it is claimed that new physical fields A_μ that interact with the fields you begin with are introduced.
- Beginning with a field ϕ, by choosing the symmetry group to be $U(1)$, ϕ picks up an interation with the classical electromagnetic field, for $SU(3)$ ϕ picks up an interaction with a 'colour field,' and for $U(1) \otimes SU(2)$ ϕ picks up an interaction with an 'electroweak field.'
- By starting with the appropriate sort of field ϕ, these (classical) field theories can be quantized to produce the quantum field theories of the standard model (QED, QCD and EWT).

Gauge Arguments

- By choosing certain symmetry groups under which the Lagrangian is invariant, it is claimed that new physical fields A_μ that interact with the fields you begin with are introduced.
- Beginning with a field ϕ, by choosing the symmetry group to be $U(1)$, ϕ picks up an interation with the classical electromagnetic field, for $SU(3)$ ϕ picks up an interaction with a 'colour field,' and for $U(1) \otimes SU(2)$ ϕ picks up an interaction with an 'electroweak field.'
- By starting with the appropriate sort of field ϕ, these (classical) field theories can be quantized to produce the quantum field theories of the standard model (QED, QCD and EWT).
- Thus, the gauge argument is supposed to show that the fields of the standard model arise "naturally" from the requirement that a given global symmetry holds locally.

1 Gauge Theories

2 Classical Electromagnetism

3 Lagrangian Mechanics of Fields
- Lagrangian Mechanics of Point Particles
- Lagrangian Mechanics of Fields
- Relativistic Field Theories

4 The Gauge Argument for a Complex Scalar Field

5 Yang-Mills Theory

Charges and Currents

The fundamental equations for electromagnetic theory are *Maxwell's Equations*. There are the two homogeneous equations:

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0,$$

and the two inhomogeneous equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J},$$

where ρ is the density of electric charge and $\mathbf{J}$ is the density of electric current.

Potentials

Since $\nabla \cdot \mathbf{B} = 0$ and for any vector field $\mathbf{f}$, $\nabla \cdot (\nabla \times \mathbf{f}) = 0$, the magnetic field can be defined in terms of a vector potential $\mathbf{A}$, such that

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Potentials

Since $\nabla \cdot \mathbf{B} = 0$ and for any vector field $\mathbf{f}$, $\nabla \cdot (\nabla \times \mathbf{f}) = 0$, the magnetic field can be defined in terms of a vector potential $\mathbf{A}$, such that

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

This enables the other homogeneous Maxwell equation to be written as

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0.$$

Potentials

Since $\nabla \cdot \mathbf{B} = 0$ and for any vector field $\mathbf{f}$, $\nabla \cdot (\nabla \times \mathbf{f}) = 0$, the magnetic field can be defined in terms of a vector potential $\mathbf{A}$, such that

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

This enables the other homogeneous Maxwell equation to be written as

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0.$$

Then, since for any scalar field f, $\nabla \times (\nabla f) = 0$, the quantity above with the vanishing curl can be written in terms of a scalar potential φ, such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \varphi.$$

Potentials

Thus, we can write the electric and magnetic fields, **E** and **B**, in terms of a vector and scalar potential, **A** and φ, as

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}.$$

Gauge Freedom

Since the magnetic field is defined as the curl of the vector potential **A**, the gradient of any scalar field Λ can be added to **A** without changing the magnetic field.

Gauge Freedom

Since the magnetic field is defined as the curl of the vector potential $\mathbf{A}$, the gradient of any scalar field Λ can be added to $\mathbf{A}$ without changing the magnetic field.

Thus, the transformation

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} - \nabla\Lambda$$

leaves the magnetic field invariant. So by describing the magnetic field in terms of a potential a descriptive freedom is introduced, since there is no unique vector potential that determines the magnetic field.

Gauge Freedom

Since the magnetic field is defined as the curl of the vector potential $\mathbf{A}$, the gradient of any scalar field Λ can be added to $\mathbf{A}$ without changing the magnetic field.

Thus, the transformation

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} - \nabla\Lambda$$

leaves the magnetic field invariant. So by describing the magnetic field in terms of a potential a descriptive freedom is introduced, since there is no unique vector potential that determines the magnetic field.

This descriptive freedom is called *gauge freedom* and the transformation of the potential leaving the field invariant is called a *gauge transformation*.

Gauge Transformations

If the vector potential is transformed as

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} - \nabla\Lambda,$$

then in order to leave the electric field invariant we must also transform ϕ as

$$\varphi \longrightarrow \varphi' = \varphi + \frac{\partial\Lambda}{\partial t}.$$

Gauge Transformations

If the vector potential is transformed as

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} - \nabla\Lambda,$$

then in order to leave the electric field invariant we must also transform ϕ as

$$\varphi \longrightarrow \varphi' = \varphi + \frac{\partial\Lambda}{\partial t}.$$

Thus, we have that under a gauge transformation

$$(\varphi, \mathbf{A}) \longrightarrow (\varphi, \mathbf{A}) + \left(\frac{\partial}{\partial t}, -\nabla\right)\Lambda.$$

Special Relativity

Maxwell's equations can be written in Lorentz covariant form. Local conservation of charge, which is expressed by the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

implies that the charge and current densities, ρ and $\mathbf{J}$, together form a 4-vector J^μ (and adopting units such that $c = 1$):

$$J^\mu = (\rho, \mathbf{J}).$$

Special Relativity

Maxwell's equations can be written in Lorentz covariant form. Local conservation of charge, which is expressed by the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

implies that the charge and current densities, ρ and $\mathbf{J}$, together form a 4-vector J^μ (and adopting units such that $c = 1$):

$$J^\mu = (\rho, \mathbf{J}).$$

Then the continuity equation can be written in a Lorentz covariant form as

$$\partial_\mu J^\mu = 0,$$

Special Relativity

Maxwell's equations can be written in Lorentz covariant form. Local conservation of charge, which is expressed by the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

implies that the charge and current densities, ρ and $\mathbf{J}$, together form a 4-vector J^μ (and adopting units such that $c = 1$):

$$J^\mu = (\rho, \mathbf{J}).$$

Then the continuity equation can be written in a Lorentz covariant form as

$$\partial_\mu J^\mu = 0,$$

given the summation convention and where ∂_μ is the covariant differential operator

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla \right).$$

Potentials Form a 4-vector

It can be shown that the scalar and vector potentials, φ and $\mathbf{A}$, together form a contravariant 4-vector

$$A^{\mu} = (\varphi, \mathbf{A}).$$

Potentials Form a 4-vector

It can be shown that the scalar and vector potentials, φ and $\mathbf{A}$, together form a contravariant 4-vector

$$A^\mu = (\varphi, \mathbf{A}).$$

Maxwell's equations can be written in a Lorentz covariant form in terms of A^μ.

Potentials Form a 4-vector

It can be shown that the scalar and vector potentials, φ and $\mathbf{A}$, together form a contravariant 4-vector

$$A^\mu = (\varphi, \mathbf{A}).$$

Maxwell's equations can be written in a Lorentz covariant form in terms of A^μ.

Thus, the gauge transformation

$$(\varphi, \mathbf{A}) \longrightarrow (\varphi, \mathbf{A}) + \left(\tfrac{\partial}{\partial t}, -\nabla\right) \Lambda$$

Potentials Form a 4-vector

It can be shown that the scalar and vector potentials, φ and $\mathbf{A}$, together form a contravariant 4-vector

$$A^\mu = (\varphi, \mathbf{A}).$$

Maxwell's equations can be written in a Lorentz covariant form in terms of A^μ.

Thus, the gauge transformation

$$(\varphi, \mathbf{A}) \longrightarrow (\varphi, \mathbf{A}) + \left(\tfrac{\partial}{\partial t}, -\nabla\right)\Lambda$$

can be written as

$$A^\mu \longrightarrow A^\mu + \partial^\mu \Lambda,$$

where $\partial^\mu = \left(\frac{\partial}{\partial t}, -\nabla\right)$ is the contravariant differential operator.

Lorentz Covariant Maxwell's Equations

Given the expressions for **E** and **B** in terms of the scalar and vector potentials above, the various components of the fields can be expressed in terms of the components of A^μ.

Lorentz Covariant Maxwell's Equations

Given the expressions for **E** and **B** in terms of the scalar and vector potentials above, the various components of the fields can be expressed in terms of the components of A^μ. For example, we have for the x components that

$$E_x = -\frac{\partial A_x}{\partial t} - \frac{\partial \varphi}{\partial x} = -(\partial^0 A^1 - \partial^1 A^0),$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2),$$

given $A^\mu = (\varphi, \mathbf{A})$ and $\partial^\mu = \left(\frac{\partial}{\partial t}, -\nabla\right)$.

Lorentz Covariant Maxwell's Equations

The six equations for the six components of **E** and **B** determine a second rank, antisymmetric field strength tensor

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}.$$

Lorentz Covariant Maxwell's Equations

The six equations for the six components of **E** and **B** determine a second rank, antisymmetric field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu .$$

Maxwell's equations can be written in Lorentz covariant form in terms of $F^{\mu\nu}$.

Lorentz Covariant Maxwell's Equations

The two inhomogeneous Maxwell equations can be written as

$$\partial_\mu F^{\mu\nu} = J^\nu,$$

with $J^\nu = (\rho, \mathbf{J})$ the 4-current.

Lorentz Covariant Maxwell's Equations

The two homogeneous Maxwell equations can be written as the four equations

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0.$$

1 Gauge Theories

2 Classical Electromagnetism

3 Lagrangian Mechanics of Fields
- Lagrangian Mechanics of Point Particles
- Lagrangian Mechanics of Fields
- Relativistic Field Theories

4 The Gauge Argument for a Complex Scalar Field

5 Yang-Mills Theory

Lagrangian Formulation of Mechanics

The Lagrangian formulation of mechanics characterizes some physical system in terms of independent *generalized coordinates* q_i and their time derivatives $\dot{q}_i$.

Lagrangian Formulation of Mechanics

The Lagrangian formulation of mechanics characterizes some physical system in terms of independent *generalized coordinates* q_i and their time derivatives $\dot{q}_i$.

The number of independent coordinates is determined by the number of degrees of freedom of the system.

Lagrangian Formulation of Mechanics

The Lagrangian formulation of mechanics characterizes some physical system in terms of independent *generalized coordinates* q_i and their time derivatives $\dot{q}_i$.

The number of independent coordinates is determined by the number of degrees of freedom of the system.

For example, an ideal pendulum has one degree of freedom and can be described in terms of the angle θ that the string makes with the equilibrium position of the string (vertical) and its time derivative $\dot{\theta}$.

Lagrangian Formulation of Mechanics

The Lagrangian formulation of mechanics characterizes some physical system in terms of independent *generalized coordinates* q_i and their time derivatives $\dot{q}_i$.

The number of independent coordinates is determined by the number of degrees of freedom of the system.

For example, an ideal pendulum has one degree of freedom and can be described in terms of the angle θ that the string makes with the equilibrium position of the string (vertical) and its time derivative $\dot{\theta}$.

The Lagrangian $L(q_i, \dot{q}_i, t) = T - V$, where T is the kinetic energy and V the potential energy of the system.

Lagrangian Formulation of Mechanics

The motion of the system from t_1 to t_2 is determined by finding the path such that the *action integral*

$$S = \int_{t_1}^{t_2} L dt$$

has a ‘stationary value.’

Lagrangian Formulation of Mechanics

The motion of the system from t_1 to t_2 is determined by finding the path such that the *action integral*

$$S = \int_{t_1}^{t_2} L dt$$

has a 'stationary value.' That is to say, the system follows the path such that small changes of that path, with t_1 and t_2 fixed, leave S unchanged.

Lagrangian Formulation of Mechanics

The motion of the system from t_1 to t_2 is determined by finding the path such that the *action integral*

$$S = \int_{t_1}^{t_2} L dt$$

has a 'stationary value.' That is to say, the system follows the path such that small changes of that path, with t_1 and t_2 fixed, leave S unchanged. This can be expressed by saying that the motion is such that the *variation* of the action integral for fixed t_1 and t_2 is 0, *i.e.*

$$\delta S = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt = 0.$$

Lagrangian Formulation of Mechanics

The motion of the system from t_1 to t_2 is determined by finding the path such that the *action integral*

$$S = \int_{t_1}^{t_2} L dt$$

has a 'stationary value.' That is to say, the system follows the path such that small changes of that path, with t_1 and t_2 fixed, leave S unchanged. This can be expressed by saying that the motion is such that the *variation* of the action integral for fixed t_1 and t_2 is 0, *i.e.*

$$\delta S = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt = 0.$$

This is called *Hamilton's Principle.*

Euler-Lagrange Equations

Given the appropriate sort of constraints on the system, such that the q_i can be treated as independent, the variational equation above can be solved to obtain the *Euler-Lagrange Equations*:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

Euler-Lagrange Equations

Given the appropriate sort of constraints on the system, such that the q_i can be treated as independent, the variational equation above can be solved to obtain the *Euler-Lagrange Equations*:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

This all yields a formulation of mechanics alternative to one founded on Newton's laws.

Symmetry and Lagrangian Mechanics

If the Lagrangian is invariant under transformations of one or more of the generalized coordinates q_i, then the system has one or more *conserved quantities*. This connection is established by Noether's theorem. Thus, symmetries of the Lagrangian give rise to conserved quantities.

Symmetry and Lagrangian Mechanics

To see how this works, suppose that $L(q_i, \dot{q}_i, t)$ does not depend on q_k. Then

$$\frac{\partial L}{\partial q_k} = 0.$$

Symmetry and Lagrangian Mechanics

To see how this works, suppose that $L(q_i, \dot{q}_i, t)$ does not depend on q_k. Then

$$\frac{\partial L}{\partial q_k} = 0.$$

Thus, the Euler-Lagrange equation for $i = k$ reduces to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0.$$

Symmetry and Lagrangian Mechanics

Then, letting

$$p_k =_{def} \frac{\partial L}{\partial \dot{q}_i},$$

we have that

$$\frac{dp_k}{dt} = 0$$

Symmetry and Lagrangian Mechanics

Then, letting

$$p_k =_{def} \frac{\partial L}{\partial \dot{q}_i},$$

we have that

$$\frac{dp_k}{dt} = 0$$

or

$$p_k = \text{constant}.$$

Symmetry and Lagrangian Mechanics

Then, letting

$$p_k =_{def} \frac{\partial L}{\partial \dot{q}_i},$$

we have that

$$\frac{dp_k}{dt} = 0$$

or

$$p_k = \text{constant}.$$

p_k is a *generalized momentum*, so invariance of L under changes in q_k implies conservation of the generalized momentum p_k.

Lagrangian Mechanics of Fields

We now shift from discrete generalized coordinates to *fields* ϕ_i,

$$\phi_i(x^\mu) = \phi_i(x^0, x^1, x^2, x^3) = \phi_i(t, x, y, z).$$

Lagrangian Mechanics of Fields

We now shift from discrete generalized coordinates to *fields* ϕ_i,

$$\phi_i(x^\mu) = \phi_i(x^0, x^1, x^2, x^3) = \phi_i(t, x, y, z).$$

The shift can be thought of as shifting from generalized coordinates $q_i(t)$ which are functions of time t, to fields $\phi_i(x^\mu)$, which are functions of spacetime location x^μ.

Lagrangian Mechanics of Fields

We shift from talking about a Lagrangian L to talking about a Lagrangian density $\mathcal{L}(\phi_i, \partial_\mu \phi_i)$.

Lagrangian Mechanics of Fields

We shift from talking about a Lagrangian L to talking about a Lagrangian density $\mathcal{L}(\phi_i, \partial_\mu \phi_i)$.The equivalent of the Lagrangian is the integral of $\mathcal{L}$ over all space:

$$L = \int \mathcal{L} d^3 x.$$

Lagrangian Mechanics of Fields

The dynamics of the system are calculated by minimizing an action integral

$$S = \int \mathcal{L} d^4 x.$$

Lagrangian Mechanics of Fields

Consider now just a single field ϕ.

Lagrangian Mechanics of Fields

Consider now just a single field ϕ. By determining the field configuration in some region R of spacetime such that the variation δS of the action integral is zero when that the value of the field on the boundary of R is fixed,

Lagrangian Mechanics of Fields

Consider now just a single field ϕ. By determining the field configuration in some region R of spacetime such that the variation δS of the action integral is zero when that the value of the field on the boundary of R is fixed, we obtain the Euler-Lagrange equations for the field:

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

Invariance of the Lagrangian Under Groups of Transformations

Following a similar but more general argument that leads to the Euler-Lagrange equations for the field, and assuming that the Lagrangian density $\mathcal{L}$ is invariant under some group of transformations of x^μ and ϕ, then it follows that there must be a *conserved current* J^μ, *i.e.* there is a J^μ such that

$$\partial_\mu J^\mu = 0.$$

Invariance of the Lagrangian Under Groups of Transformations

Following a similar but more general argument that leads to the Euler-Lagrange equations for the field, and assuming that the Lagrangian density $\mathcal{L}$ is invariant under some group of transformations of x^μ and ϕ, then it follows that there must be a *conserved current* J^μ, *i.e.* there is a J^μ such that

$$\partial_\mu J^\mu = 0.$$

This entails the existence of a *conserved charge* Q, which for some time $t =$ constant,

$$Q = \int_V J^0 d^3x,$$

where V is a 3-volume in the spacelike hypersurface at time t.

Invariance of the Lagrangian Under Groups of Transformations

That the existence of a conserved current J^μ and charge Q is entailed by the invariance of the Lagrangian density under the (not here specified) group of transformations is the content of Noether's theorem in the present context.

Invariance of the Lagrangian Under Groups of Transformations

That the existence of a conserved current J^{μ} and charge Q is entailed by the invariance of the Lagrangian density under the (not here specified) group of transformations is the content of Noether's theorem in the present context.

Invariance of the Lagrangian density under translation of the origin of space and time lead to conservation of momentum and energy, respectively, and invariance under spatial rotations leads to conservation of angular momentum.

Klein-Gordon Equation

It is possible to motivate the Schrödinger equation by starting with the classical energy-momentum equation

$$\frac{\mathbf{p}^2}{2m} + V = E$$

Klein-Gordon Equation

It is possible to motivate the Schrödinger equation by starting with the classical energy-momentum equation

$$\frac{\mathbf{p}^2}{2m} + V = E$$

and make the substitutions

$$\mathbf{p} \longrightarrow \frac{\hbar}{i}\nabla, \qquad E \longrightarrow i\hbar\frac{\partial}{\partial t}$$

Klein-Gordon Equation

It is possible to motivate the Schrödinger equation by starting with the classical energy-momentum equation

$$\frac{\mathbf{p}^2}{2m} + V = E$$

and make the substitutions

$$\mathbf{p} \longrightarrow \frac{\hbar}{i}\nabla, \qquad E \longrightarrow i\hbar\frac{\partial}{\partial t}$$

to give

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

by letting the operators act on a wave function ψ.

Klein-Gordon Equation

If, on the other hand, we are seeking compatibility with special relativity, we might start with the relativistic energy-momentum equation

$$E^2 - \mathbf{p}^2 c^2 = m^2 c^2,$$

Klein-Gordon Equation

If, on the other hand, we are seeking compatibility with special relativity, we might start with the relativistic energy-momentum equation

$$E^2 - \mathbf{p}^2 c^2 = m^2 c^2,$$

which in Lorentz covariant form is

$$p_\mu p^\mu = m^2 c^2.$$

Klein-Gordon Equation

Following the same prescription we make the substitution

$$p_\mu \longrightarrow i\hbar\partial_\mu,$$

Klein-Gordon Equation

Following the same prescription we make the substitution

$$p_\mu \longrightarrow i\hbar\partial_\mu,$$

which yields

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2\right)\psi=\left(\frac{mc}{\hbar}\right)^2\psi$$

by letting the operators act on a wave function ψ.

Klein-Gordon Equation

Following the same prescription we make the substitution

$$p_\mu \longrightarrow i\hbar\partial_\mu,$$

which yields

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\psi = \left(\frac{mc}{\hbar}\right)^2 \psi$$

by letting the operators act on a wave function ψ. Notice that the operator on the left hand side is $-\partial_\mu\partial^\mu$.

Klein-Gordon Equation

Following the same prescription we make the substitution

$$p_\mu \longrightarrow i\hbar\partial_\mu,$$

which yields

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\psi = \left(\frac{mc}{\hbar}\right)^2 \psi$$

by letting the operators act on a wave function ψ. Notice that the operator on the left hand side is $-\partial_\mu\partial^\mu$. Thus, letting $\Box =_{def} \partial_\mu\partial^\mu$ (and setting $\hbar = 1$ and $c = 1$), the above equation reads

$$\Box\psi + m^2\psi = 0.$$

Klein-Gordon Equation

Following the same prescription we make the substitution

$$p_\mu \longrightarrow i\hbar\partial_\mu,$$

which yields

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\psi = \left(\frac{mc}{\hbar}\right)^2 \psi$$

by letting the operators act on a wave function ψ. Notice that the operator on the left hand side is $-\partial_\mu\partial^\mu$. Thus, letting $\Box =_{def} \partial_\mu\partial^\mu$ (and setting $\hbar = 1$ and $c = 1$), the above equation reads

$$\Box\psi + m^2\psi = 0.$$

This is called the *Klein-Gordon equation*.

Klein-Gordon Equation

In quantum field theory the Klein-Gordon Equation describes a spin-0 quantum field ϕ, the particles of which have mass m.

Klein-Gordon Equation

In quantum field theory the Klein-Gordon Equation describes a spin-0 quantum field ϕ, the particles of which have mass m. For a real field ϕ it can be derived from the Lagrangian density (hence forward Lagrangian)

$$\mathcal{L} = \frac{1}{2}\eta^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi) - \frac{m^2}{2}\phi^2,$$

where $\eta^{\alpha\beta}$ is the Minkowski metric.

1. Gauge Theories
2. Classical Electromagnetism
3. Lagrangian Mechanics of Fields
   - Lagrangian Mechanics of Point Particles
   - Lagrangian Mechanics of Fields
   - Relativistic Field Theories
4. The Gauge Argument for a Complex Scalar Field
5. Yang-Mills Theory

An Example of A Gauge Argument

Let us consider a field ϕ that takes complex values. We shall develop an instance of the gauge argument by considering a 'Klein-Gordon' Lagrangian for this field.

An Example of A Gauge Argument

Let us consider a field ϕ that takes complex values. We shall develop an instance of the gauge argument by considering a 'Klein-Gordon' Lagrangian for this field.

A complex scalar field has two real parts ϕ_1 and ϕ_2. Thus, we may set

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$
$$\phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

Lagrangian for a Complex Scalar Field

Given the modified Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m\phi\phi^*,$$

the Euler-Lagrange equations yield two Klein-Gordon equations

$$(\Box + m^2)\psi = 0,$$

$$(\Box + m^2)\psi^* = 0.$$

'Global' Gauge Invariance

The Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m\phi\phi^*,$$

is easily seen to be invariant under the constant phase transformation

$$\phi \longrightarrow e^{-i\Lambda}\phi, \quad \phi^* \longrightarrow e^{i\Lambda}\phi^*,$$

where Λ is a real constant.

'Global' Gauge Invariance

The Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m\phi\phi^*,$$

is easily seen to be invariant under the constant phase transformation

$$\phi \longrightarrow e^{-i\Lambda}\phi, \quad \phi^* \longrightarrow e^{i\Lambda}\phi^*,$$

where Λ is a real constant. This is called a *global gauge transformation* (or *gauge transformation of the first kind*).

'Global' Gauge Invariance

The Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m\phi\phi^*,$$

is easily seen to be invariant under the constant phase transformation

$$\phi \longrightarrow e^{-i\Lambda}\phi, \quad \phi^* \longrightarrow e^{i\Lambda}\phi^*,$$

where Λ is a real constant. This is called a *global gauge transformation* (or *gauge transformation of the first kind*). The term *constant* gauge transformation is perhaps more appropriate, constant since Λ is a constant (it does not depend on spacetime location).

Noether Current and Charge

The invariance of the Lagrangian under this constant gauge transformation gives rise (via Noether's theorem) to a conserved current J^μ, *i.e.* a current satisfying

$$\partial_\mu J^\mu = 0$$

Noether Current and Charge

The invariance of the Lagrangian under this constant gauge transformation gives rise (via Noether's theorem) to a conserved current J^μ, *i.e.* a current satisfying

$$\partial_\mu J^\mu = 0$$

and a conserved charge

$$Q = \int_V J^0 dV,$$

i.e. $Q =$ constant.

Gauge Transformation as an Internal Rotation

The constant gauge transformation described above can be thought of geometrically. Letting

$$\vec{\phi} = \phi_1 \hat{i} + \phi_2 \hat{j}$$

the gauge transformation can be thought of as a rotation of the vector $\vec{\phi}$ through an angle Λ.

Gauge Transformation as an Internal Rotation

The constant gauge transformation described above can be thought of geometrically. Letting

$$\vec{\phi} = \phi_1 \hat{i} + \phi_2 \hat{j}$$

the gauge transformation can be thought of as a rotation of the vector $\vec{\phi}$ through an angle Λ. The rotation is represented by the 1×1 complex matrix $e^{i\Lambda}$.

Gauge Transformation as an Internal Rotation

The constant gauge transformation described above can be thought of geometrically. Letting

$$\vec{\phi} = \phi_1 \hat{i} + \phi_2 \hat{j}$$

the gauge transformation can be thought of as a rotation of the vector $\vec{\phi}$ through an angle Λ. The rotation is represented by the 1×1 complex matrix $e^{i\Lambda}$. Since the Lagrangian is invariant under all such matrices, the Lagrangian is invariant under the group $U(1)$.

Making the Gauge Symmetry 'Local'

The next stage in the gauge argument is to make the 'global', or constant, transformation 'local', or *variable*.

Making the Gauge Symmetry 'Local'

The next stage in the gauge argument is to make the 'global', or constant, transformation 'local', or *variable*. This is to say, an attempt is made to make the Lagrangian invariant under internal rotations $e^{i\Lambda(x^\mu)}$, where Λ is now a function of spacetime location x^μ (such a transformation is also called a *gauge transformation of the second kind*).

Making the Gauge Symmetry 'Local'

The next stage in the gauge argument is to make the 'global', or constant, transformation 'local', or *variable*. This is to say, an attempt is made to make the Lagrangian invariant under internal rotations $e^{i\Lambda(x^\mu)}$, where Λ is now a function of spacetime location x^μ (such a transformation is also called a *gauge transformation of the second kind*). Thought of geometrically, the vector

$$\vec{\phi}(x^\mu) = \phi(x^\mu)\hat{i} + \phi^*(x^\mu)\hat{j}$$

is (in general) rotated by a different angle $\Lambda(x^\mu)$ at each spacetime location x^μ (and in such a way that the amount of rotation changes smoothly from spacetime point to spacetime point).

Justifying Making the Gauge Symmetry 'Local'

Physicists try to justify this move to make the gauge symmetry local. Ryder (1996) says that

> *So [for a global gauge transformation] when we perform a rotation in the internal space of $\vec{\phi}$ at one point, through an angle of Λ, we must perform the same rotation at all other points at the same time. If we take this physical interpretation seriously, we see that it is impossible to fulfil, since it contradicts the letter and spirit of relativity, according to which there must be a minimum time delay equal to the time of light travel. To get round this problem we simply abandon the requirement that Λ is a constant, and write it as an arbitrary function of space-time, $\Lambda(x^\mu)$. (93)*

Is this Justification Acceptable?

Healey criticizes this argument for the following reasons:

Is this Justification Acceptable?

Healey criticizes this argument for the following reasons:

- This argument cannot work since a constant gauge transformation is just a special case of a variable one; and

Is this Justification Acceptable?

Healey criticizes this argument for the following reasons:

- This argument cannot work since a constant gauge transformation is just a special case of a variable one; and
- Relativity does not cause such a problem, since invariance of the field ϕ under a constant change in phase, *i.e.* under a constant gauge transformation, does not require that such a transformation be carried out physically.

Is this Justification Acceptable?

Healey criticizes this argument for the following reasons:

- This argument cannot work since a constant gauge transformation is just a special case of a variable one; and
- Relativity does not cause such a problem, since invariance of the field ϕ under a constant change in phase, *i.e.* under a constant gauge transformation, does not require that such a transformation be carried out physically. The representation of the field is only determined up to a constant overall phase, making constant gauge invariance a theoretical symmetry rather than an empirical one.

Is this Justification Acceptable?

Healey criticizes this argument for the following reasons:

- This argument cannot work since a constant gauge transformation is just a special case of a variable one; and
- Relativity does not cause such a problem, since invariance of the field ϕ under a constant change in phase, *i.e.* under a constant gauge transformation, does not require that such a transformation be carried out physically. The representation of the field is only determined up to a constant overall phase, making constant gauge invariance a theoretical symmetry rather than an empirical one.
- Healey points out that constant phase invariance is not an empirical symmetry because only phase relations between two distinct fields are observable. The empirical content of constant gauge invariance is in the conserved Noether current and charge.

A Better Justification for Variable Gauge Invariance?

Healey cites a better justification of the move 'from global to local' gauge invariance (attributed to Auyang (1995) who follows Weyl (1929)) as

> *an abandonment of the assumption that meaningful comparisons even of relative phases may be made without adoption of some prior convention as to what is to count as the same phase at different space-time points. (162)*

A Better Justification for Variable Gauge Invariance?

Healey cites a better justification of the move 'from global to local' gauge invariance (attributed to Auyang (1995) who follows Weyl (1929)) as

> *an abandonment of the assumption that meaningful comparisons even of relative phases may be made without adoption of some prior convention as to what is to count as the same phase at different space-time points. (162)*

Regarding the choice of phase at different space-time points as conventional is naturally accommodated by the fibre bundle formulation, but this just makes a choice of variable gauge a choice among one of many equivalent ways of representing the same matter field. There are no empirical implications of this choice of gauge.

Loss of Lorentz and Gauge Invariance

Let us continue with the gauge argument. . .

Loss of Lorentz and Gauge Invariance

Let us continue with the gauge argument. . .

Making the gauge transformation variable causes the transformed Lagrangian to fail to be Lorentz covariant and the Lagrangian, as it stands, is not invariant under this variable gauge transformation.

Loss of Lorentz and Gauge Invariance

Let us continue with the gauge argument...

Making the gauge transformation variable causes the transformed Lagrangian to fail to be Lorentz covariant and the Lagrangian, as it stands, is not invariant under this variable gauge transformation. The change $\delta\mathcal{L}$ of the Lagrangian under the variable transformation is

$$\delta\mathcal{L} = (\partial_\mu \Lambda) J^\mu,$$

where J^μ is the Noether current.

Attempt to Restore Gauge Invariance

To restore invariance under the variable transformation, a new 4-vector field A_μ that couples directly to the current J^μ is introduced, which adds an additional term to the Lagrangian $\mathcal{L}$:

$$\mathcal{L}_1 = -J^\mu A_\mu.$$

Attempt to Restore Gauge Invariance

To restore invariance under the variable transformation, a new 4-vector field A_μ that couples directly to the current J^μ is introduced, which adds an additional term to the Lagrangian $\mathcal{L}$:

$$\mathcal{L}_1 = -J^\mu A_\mu.$$

It is also required that

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda$$

under a variable gauge transformation.

Attempt to Restore Gauge Invariance

To restore invariance under the variable transformation, a new 4-vector field A_μ that couples directly to the current J^μ is introduced, which adds an additional term to the Lagrangian $\mathcal{L}$:

$$\mathcal{L}_1 = -J^\mu A_\mu.$$

It is also required that

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda$$

under a variable gauge transformation. Notice that this is of the same form as the gauge transformation of the electromagnetic potential A_μ.

Attempt to Restore Gauge Invariance

Under a variable gauge transformation this new term and new vector field that transforms as described produces a term that cancels the term $\delta\mathcal{L}$ above, but produces an additional term so that

$$\delta\mathcal{L} + \delta L_1 = -2A_\mu(\partial^\mu)\phi^*\phi.$$

Attempt to Restore Gauge Invariance

Under a variable gauge transformation this new term and new vector field that transforms as described produces a term that cancels the term $\delta\mathcal{L}$ above, but produces an additional term so that

$$\delta\mathcal{L} + \delta L_1 = -2A_\mu(\partial^\mu)\phi^*\phi.$$

Thus, an additional term is added to the Lagrangian:

$$\mathcal{L}_2 = A_\mu A^\mu \phi^* \phi,$$

which finally restores invariance,

Attempt to Restore Gauge Invariance

Under a variable gauge transformation this new term and new vector field that transforms as described produces a term that cancels the term $\delta\mathcal{L}$ above, but produces an additional term so that

$$\delta\mathcal{L} + \delta L_1 = -2A_\mu(\partial^\mu)\phi^*\phi.$$

Thus, an additional term is added to the Lagrangian:

$$\mathcal{L}_2 = A_\mu A^\mu \phi^* \phi,$$

which finally restores invariance, *i.e.*

$$\delta\mathcal{L} + \delta L_1 + \delta L_2 = 0.$$

Implications of Restoration of Invariance

Thus, the Lagrangian

$$\mathcal{L} + \mathcal{L}_1 + \mathcal{L}_2 = (D_\mu \phi)(D^\mu \phi^*) - m^2 \phi \phi^*,$$

where $D_\mu = \partial_\mu + iA_\mu$ is the *covariant derivative operator*, is invariant under variable gauge transformations.

Implications of Restoration of Invariance

Thus, the Lagrangian

$$\mathcal{L} + \mathcal{L}_1 + \mathcal{L}_2 = (D_\mu \phi)(D^\mu \phi^*) - m^2 \phi\phi^*,$$

where $D_\mu = \partial_\mu + iA_\mu$ is the *covariant derivative operator*, is invariant under variable gauge transformations. This Lagrangian has the same form as the Klein-Gordon Lagrangian we started with except that the ordinary derivative operator is replaced by the covariant derivative operator.

Implications of Restoration of Invariance

Thus, the Lagrangian

$$\mathcal{L} + \mathcal{L}_1 + \mathcal{L}_2 = (D_\mu \phi)(D^\mu \phi^*) - m^2 \phi \phi^*,$$

where $D_\mu = \partial_\mu + iA_\mu$ is the *covariant derivative operator*, is invariant under variable gauge transformations. This Lagrangian has the same form as the Klein-Gordon Lagrangian we started with except that the ordinary derivative operator is replaced by the covariant derivative operator.

We now see that as a result of demanding local gauge invariance, so the argument goes, we have had to introduce a new vector field A_μ that couples to the current J^μ of the complex field ϕ.

Questioning the Introduction of a 'New Field'

Since, in the usual form of the gauge argument, the field A_μ is regarded as a new physical field, it is natural to make the move to introduce a constant e along with the new field, *i.e.* to introduce eA_μ rather than A_μ as we have done, where e is the coupling constant between the field A_μ and the current J^μ.

Questioning the Introduction of a 'New Field'

Since, in the usual form of the gauge argument, the field A_μ is regarded as a new physical field, it is natural to make the move to introduce a constant e along with the new field, *i.e.* to introduce eA_μ rather than A_μ as we have done, where e is the coupling constant between the field A_μ and the current J^μ.

Healey and others point out that there is no good reason at this point, beyond that of a suggestive heuristic, to regard the 4-vector A_μ as a *physical* field, *i.e.* to think that $A_\mu \neq 0$. It may just be an artefact of extending the theory of the field ϕ to the case where there is an arbitrary choice of variable phase.

Completion of the Gauge Argument

If, however, A_μ is regarded as a new physical field, then the last part of the gauge argument is that the vector field A_μ ought to contribute directly to the Lagrangian.

Completion of the Gauge Argument

If, however, A_μ is regarded as a new physical field, then the last part of the gauge argument is that the vector field A_μ ought to contribute directly to the Lagrangian. Thus, a gauge invariant term depending only on A_μ is sought. It is seen that the 4-dimensional curl of A_μ

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is gauge invariant.

Completion of the Gauge Argument

If, however, A_μ is regarded as a new physical field, then the last part of the gauge argument is that the vector field A_μ ought to contribute directly to the Lagrangian. Thus, a gauge invariant term depending only on A_μ is sought. It is seen that the 4-dimensional curl of A_μ

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is gauge invariant. Then, the term

$$\mathcal{L}_3 = -\lambda F^{\mu\nu} F_{\mu\nu}$$

is both gauge invariant and Lorentz covariant, and is added to the Lagrangian.

Completion of the Gauge Argument

Notice that the fact that $F_{\mu\nu}$ and A_μ are related by the equation

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

makes A_μ look a lot like the vector potential from electromagnetic theory and makes $F_{\mu\nu}$ looks a lot like the electromagnetic field tensor!

Completion of the Gauge Argument

Notice that the fact that $F_{\mu\nu}$ and A_μ are related by the equation

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

makes A_μ look a lot like the vector potential from electromagnetic theory and makes $F_{\mu\nu}$ looks a lot like the electromagnetic field tensor! (Of course, the choice of notation helps. . .).

Completion of the Gauge Argument

The inclination to identify A_μ with the vector potential and $F_{\mu\nu}$ with the electromagnetic field tensor is strengthened by the following fact.

Completion of the Gauge Argument

The inclination to identify A_μ with the vector potential and $F_{\mu\nu}$ with the electromagnetic field tensor is strengthened by the following fact. If the action integral with the new Lagrangian

$$\mathcal{L}_{\text{tot}} = \mathcal{L} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$$

is determined to be stationary under variation of A_μ, then the Euler-Lagrange equations yield

$$\partial_\nu F^{\mu\nu} = \mathcal{J}^\mu,$$

where $\mathcal{J}^\mu$ is a 'covariant' version of J^μ. These are identical in form to the inhomogeneous Maxwell equations.

Completion of the Gauge Argument

The inclination to identify A_μ with the vector potential and $F_{\mu\nu}$ with the electromagnetic field tensor is strengthened by the following fact. If the action integral with the new Lagrangian

$$\mathcal{L}_{\text{tot}} = \mathcal{L} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$$

is determined to be stationary under variation of A_μ, then the Euler-Lagrange equations yield

$$\partial_\nu F^{\mu\nu} = \mathcal{J}^\mu,$$

where $\mathcal{J}^\mu$ is a ‘covariant’ version of J^μ. These are identical in form to the inhomogeneous Maxwell equations. It follows from this equation, that

$$\partial_\mu \mathcal{J}^\mu = 0,$$

i.e. the current $\mathcal{J}^\mu$ is conserved.

Completion of the Gauge Argument

The inclination to identify A_μ with the vector potential and $F_{\mu\nu}$ with the electromagnetic field tensor is strengthened by the following fact. If the action integral with the new Lagrangian

$$\mathcal{L}_{\text{tot}} = \mathcal{L} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$$

is determined to be stationary under variation of A_μ, then the Euler-Lagrange equations yield

$$\partial_\nu F^{\mu\nu} = \mathcal{J}^\mu,$$

where $\mathcal{J}^\mu$ is a 'covariant' version of J^μ. These are identical in form to the inhomogeneous Maxwell equations. It follows from this equation, that

$$\partial_\mu \mathcal{J}^\mu = 0,$$

i.e. the current $\mathcal{J}^\mu$ is conserved. Thus, it is the current $\mathcal{J}^\mu$ that is conserved when the field $F_{\mu\nu}$ is present.

Completion of the Gauge Argument

Thus, according to the usual gauge argument, as a result of the preceding argument it is concluded that the insistence that the Lagrangian be invariant under variable $U(1)$ gauge transformations requires the introduction of fields A_μ and $F_{\mu\nu}$, which are precisely the electromagnetic vector potential and field tensor respectively.

Completion of the Gauge Argument

Thus, according to the usual gauge argument, as a result of the preceding argument it is concluded that the insistence that the Lagrangian be invariant under variable $U(1)$ gauge transformations requires the introduction of fields A_μ and $F_{\mu\nu}$, which are precisely the electromagnetic vector potential and field tensor respectively.

If eA_μ and $eF_{\mu\nu}$ are introduced rather than A_μ and $F_{\mu\nu}$ as we have done, and we set $\lambda = -\frac{1}{4e^2}$, then the Lagrangian $\mathcal{L}_{\text{tot}}$ is precisely the Lagrangian for a complex scalar field interacting with the electromagnetic field.

Triumph of the Gauge Argument?

Thus, the claim is that the demand of ‘local’ gauge invariance requires the introduction of interaction of the complex scalar field with the electromagnetic field.

Triumph of the Gauge Argument?

Thus, the claim is that the demand of 'local' gauge invariance requires the introduction of interaction of the complex scalar field with the electromagnetic field.

If the same pattern of reasoning is applied to a Dirac field, *i.e.* starting from the Dirac Lagrangian for a spin-$\frac{1}{2}$ field, rather than a Klein-Gordon (spin-0) field, then $\mathcal{L}_{\text{tot}}$ ends up being the Lagrangian density for quantum electrodynamics.

Questioning the Gauge Argument

The last part of the gauge argument, *i.e.* the part that 'naturally' leads to the introduction of the electromagnetic field, has a weakness.

Questioning the Gauge Argument

The last part of the gauge argument, *i.e.* the part that 'naturally' leads to the introduction of the electromagnetic field, has a weakness.

- The term $\mathcal{L}_3$ is not the only term that can be added to the Lagrangian that depends only on A_μ and is gauge and Lorentz invariant.

Questioning the Gauge Argument

Healey points out, however, that it has been shown by O'Raifeartaigh (1979) that just introducing the term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ "yields the simplest, renormalizable, Lorentz- and "locally" gauge-invariant Lagrangian yielding the second-order equations of motion for the coupled system." (166-167)

Questioning the Gauge Argument

Healey points out, however, that it has been shown by O'Raifeartaigh (1979) that just introducing the term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ "yields the simplest, renormalizable, Lorentz- and "locally" gauge-invariant Lagrangian yielding the second-order equations of motion for the coupled system." (166-167) Healey adds that

> *So while the presence of this lacuna further undermines the soundness of the gauge argument, it does little to weaken the associated explanation of the properties of electromagnetism. (167)*

Questioning the Gauge Argument

Healey points out, however, that it has been shown by O'Raifeartaigh (1979) that just introducing the term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ "yields the simplest, renormalizable, Lorentz- and "locally" gauge-invariant Lagrangian yielding the second-order equations of motion for the coupled system." (166-167) Healey adds that

> *So while the presence of this lacuna further undermines the soundness of the gauge argument, it does little to weaken the associated explanation of the properties of electromagnetism. (167)*

This argument does not work for QCD, however...

Status of the Gauge Argument

Healey's overall comment on the gauge argument is the following:

> *. . . while the gauge argument effects a significant explanatory unification among the properties of diverse fundamental interactions, it certainly does not dictate their very existence. And while observations of charge conservation may yield indirect support for an empirical constant phase symmetry of matter fields, the gauge argument neither rests on nor entails a principle of "local" gauge symmetry with any empirical import, direct or indirect. "Local" gauge symmetry is a theoretical, not an empirical, symmetry. It is merely a feature of the way gauge theories of electromagnetic, electroweak, and strong interactions are conventionally formulated. (167)*

1. Gauge Theories

2. Classical Electromagnetism

3. Lagrangian Mechanics of Fields
   - Lagrangian Mechanics of Point Particles
   - Lagrangian Mechanics of Fields
   - Relativistic Field Theories

4. The Gauge Argument for a Complex Scalar Field

5. Yang-Mills Theory

Extension of the Gauge Argument

Yang and Mills extended the gauge argument to examine fields that are invariant under larger groups of internal transformations.

Extension of the Gauge Argument

Yang and Mills extended the gauge argument to examine fields that are invariant under larger groups of internal transformations. Yang and Mills considered a field with three real components that is invariant under $SU(2)$.

Extension of the Gauge Argument

Yang and Mills extended the gauge argument to examine fields that are invariant under larger groups of internal transformations. Yang and Mills considered a field with three real components that is invariant under $SU(2)$. This introduces new difficulties, since it is a non-abelian group.

Yang-Mills Field

Using a gauge argument, the requirement that the Lagrangian be invariant under ‘local’, *i.e.* variable, $SU(2)$ transformations requires the introduction of a new field $\mathbf{W}_\mu$, which is the analogue of A_μ, but has three 4-vector components.

Yang-Mills Field

Using a gauge argument, the requirement that the Lagrangian be invariant under 'local', *i.e.* variable, $SU(2)$ transformations requires the introduction of a new field $\mathbf{W}_\mu$, which is the analogue of A_μ, but has three 4-vector components.

The analogue of $F_{\mu\nu}$ is a 3-vector with second rank tensor components $\mathbf{W}_{\mu\nu}$

Yang-Mills Field

Using a gauge argument, the requirement that the Lagrangian be invariant under 'local', *i.e.* variable, $SU(2)$ transformations requires the introduction of a new field $\mathbf{W}_\mu$, which is the analogue of A_μ, but has three 4-vector components.

The analogue of $F_{\mu\nu}$ is a 3-vector with second rank tensor components $\mathbf{W}_{\mu\nu}$, which is obtained from $\mathbf{W}_\mu$ according to an equation like that of $F_{\mu\nu}$ in terms of A_μ in the $U(1)$ case but with an additional term

$$g\mathbf{W}_\mu \times \mathbf{W}_\nu .$$

Yang-Mills Field

Using a gauge argument, the requirement that the Lagrangian be invariant under 'local', *i.e.* variable, $SU(2)$ transformations requires the introduction of a new field $\mathbf{W}_\mu$, which is the analogue of A_μ, but has three 4-vector components.

The analogue of $F_{\mu\nu}$ is a 3-vector with second rank tensor components $\mathbf{W}_{\mu\nu}$, which is obtained from $\mathbf{W}_\mu$ according to an equation like that of $F_{\mu\nu}$ in terms of A_μ in the $U(1)$ case but with an additional term

$$g\mathbf{W}_\mu \times \mathbf{W}_\nu.$$

It is that the symmetry group is non-abelian that leads to the introduction of this term. This term has interesting implications, which includes the fact that the gauge field $\mathbf{W}_{\mu\nu}$ is self-interacting, *i.e.* it is a source for itself.

Too Bad for Yang and Mills, However. . .

It turns out that the theory developed by Yang and Mills is not instantiated in nature. Their method of how to develop the gauge argument for a non-abelian symmetry group G has been generalized and has found application, however.

Too Bad for Yang and Mills, However...

It turns out that the theory developed by Yang and Mills is not instantiated in nature. Their method of how to develop the gauge argument for a non-abelian symmetry group G has been generalized and has found application, however.

- For $G = SU(3)$, the classical field theory that is quantized to yield quantum chromodynamics can be obtained;

Too Bad for Yang and Mills, However…

It turns out that the theory developed by Yang and Mills is not instantiated in nature. Their method of how to develop the gauge argument for a non-abelian symmetry group G has been generalized and has found application, however.

- For $G = SU(3)$, the classical field theory that is quantized to yield quantum chromodynamics can be obtained;
- For $G = U(1) \otimes SU(2)$, the classical field theory used to develop electroweak theory can be obtained.

Too Bad for Yang and Mills, However...

It turns out that the theory developed by Yang and Mills is not instantiated in nature. Their method of how to develop the gauge argument for a non-abelian symmetry group G has been generalized and has found application, however.

- For $G = SU(3)$, the classical field theory that is quantized to yield quantum chromodynamics can be obtained;
- For $G = U(1) \otimes SU(2)$, the classical field theory used to develop electroweak theory can be obtained.

In case that $G = U(1)$ the classical field theory that when quantized yields quantum electrodynamics can obtained, which becomes a particular case of the general Yang-Mills argument.

Too Bad for Yang and Mills, However...

It turns out that the theory developed by Yang and Mills is not instantiated in nature. Their method of how to develop the gauge argument for a non-abelian symmetry group G has been generalized and has found application, however.

- For $G = SU(3)$, the classical field theory that is quantized to yield quantum chromodynamics can be obtained;
- For $G = U(1) \otimes SU(2)$, the classical field theory used to develop electroweak theory can be obtained.

In case that $G = U(1)$ the classical field theory that when quantized yields quantum electrodynamics can obtained, which becomes a particular case of the general Yang-Mills argument. For this reason the quantum field theories of the standard model can all be considered *Yang-Mills theories*.

References

- Baez, John and Javier P. Muniain. *Gauge Fields, Knots and Gravity*. New Jersey: World Scientific, 1994.
- Goldstein, Herbert, Charles Poole and John Safko. *Classical Mechanics*. 3rd Ed. Toronto: Addison Wesley, 2002.
- Griffiths, David. *Introduction to Elementary Particles*. New York: John Wiley & Sons, 1987.
- Healey, Richard. *Gauging What's Real: The Conceptual Foundations of Contemporary Gauge Theories*. New York: Oxford University Press, 2007.
- Jackson, John David. *Classical Electrodynamics*. New York: John Wiley & Sons, 1999.
- Ryder, Lewis H. *Quantum Field Theory*. 2nd Ed. Cambridge: Cambridge University Press, 1996.